

# An inverse problem for the wave equation with one measurement and the pseudorandom noise

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**Abstract.** We consider the wave equation  $(\partial_t^2 - \Delta_g)u(t, x) = f(t, x)$ , in  $\mathbb{R}^n$ ,  $u|_{\mathbb{R}_- \times \mathbb{R}^n} = 0$ , where the metric  $g = (g_{jk}(x))_{j,k=1}^n$  is known outside an open and bounded set  $M \subset \mathbb{R}^n$  with smooth boundary  $\partial M$ . We define a deterministic source  $f(t, x)$  called the pseudorandom noise as a sum of point sources,  $f(t, x) = \sum_{j=1}^{\infty} a_j \delta_{x_j}(x) \delta(t)$ , where the points  $x_j$ ,  $j \in \mathbb{Z}_+$ , form a dense set on  $\partial M$ . We show that when the weights  $a_j$  are chosen appropriately,  $u|_{\mathbb{R} \times \partial M}$  determines the scattering relation on  $\partial M$ , that is, it determines for all geodesics which pass through  $M$  the travel times together with the entering and exit points and directions. The wave  $u(t, x)$  contains the singularities produced by all point sources, but when  $a_j = \lambda^{-\lambda_j}$  for some  $\lambda > 1$ , we can trace back the point source that produced a given singularity in the data. This gives us the distance in  $(\mathbb{R}^n, g)$  between a source point  $x_j$  and an arbitrary point  $y \in \partial M$ . In particular, if  $(\overline{M}, g)$  is a simple Riemannian manifold and  $g$  is conformally Euclidian in  $\overline{M}$ , these distances are known to determine the metric  $g$  in  $M$ . In the case when  $(\overline{M}, g)$  is non-simple we present a more detailed analysis of the wave fronts yielding the scattering relation on  $\partial M$ .

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# 1 Introduction

In this paper we consider an inverse problem for the wave equation

$$\begin{aligned} (\partial_t^2 - \Delta_g)u(t, x) &= f(t, x) \quad \text{in } (0, \infty) \times \mathbb{R}^n, \\ u|_{t=0} &= \partial_t u|_{t=0} = 0, \end{aligned}$$

where  $\Delta_g$  is the Laplace-Beltrami operator corresponding to a Riemannian metric  $g(x) = [g_{jk}(x)]_{j,k=1}^n$ , that is

$$\Delta_g u = \sum_{j,k=1}^n |g|^{-1/2} \frac{\partial}{\partial x^j} \left( |g|^{1/2} g^{jk} \frac{\partial}{\partial x^k} u \right),$$

where  $|g| = \det(g_{jk})$  and  $[g^{jk}]_{j,k=1}^n = g(x)^{-1}$  is the inverse matrix of  $[g_{jk}(x)]_{j,k=1}^n$ .

We assume that  $g_{jk} \in C^\infty(\mathbb{R}^n)$  and that there are  $c_1, c_2 > 0$  such that

$$c_1 |\xi|^2 \leq \sum_{j,k=1}^n g_{jk}(x) \xi^j \xi^k \leq c_2 |\xi|^2, \quad x, \xi \in \mathbb{R}^n. \quad (1)$$

Moreover, we assume that the metric  $g$  is known outside an open and bounded set  $M \subset \mathbb{R}^n$  having a  $C^\infty$  smooth boundary  $\partial M$ .

Denote by  $d_M(x, y) = d_{M,g}(x, y)$ ,  $x, y \in \overline{M}$ , the distance function of Riemannian manifold  $(\overline{M}, g)$ , where  $g$  is considered as its restriction to  $\overline{M}$ . Let  $T > \text{diam}(M)$ , where  $\text{diam}(M) = \max\{d_M(x, y); x, y \in \overline{M}\}$ .

We choose the origin of the time axis so that the source  $f$  is active at time  $t = 0$ . To ensure compatibility with the the initial conditions we let  $T_0 < 0$  and define the measurement map  $L = L_g$ ,

$$L : C_c^\infty(T_0, T) \otimes C_c^\infty(\mathbb{R}^n) \rightarrow C^\infty((T_0, T) \times \partial M), \quad Lf = u|_{(T_0, T) \times \partial M}, \quad (2)$$

where  $u$  is the solution of the wave equation

$$\begin{aligned} (\partial_t^2 - \Delta_g)u(t, x) &= f(t, x) \quad \text{in } (T_0, T) \times \mathbb{R}^n, \\ u|_{t=T_0} &= \partial_t u|_{t=T_0} = 0. \end{aligned} \quad (3)$$

Above,  $C_c^\infty(T_0, T)$  denotes the space of smooth functions having compact support in  $(T_0, T)$ . Its dual space, the space of generalized functions or distributions, is denoted by  $\mathcal{D}'(T_0, T)$ . Moreover, for functions  $\phi \in C_c^\infty(T_0, T)$

and  $\psi \in C_c^\infty(\mathbb{R}^n)$  we denote their pointwise product by  $(\phi \otimes \psi)(t, x) = \phi(t)\psi(x)$ .

We remark that the assumption (1) together with the finite speed of propagation for the wave equation imply that the measurement  $Lf$  does not depend on  $g_{jk}(x)$ , for  $|x| > R$ , when  $R$  is sufficiently large. Thus we may assume without loss of generality that all the partial derivatives  $\partial_x^\alpha g_{jk}$  are bounded on  $\mathbb{R}^n$ .

Let  $x_j \in \partial M$ ,  $j = 1, 2, \dots$ , be a dense sequence of points in  $\partial M$ , and let us consider point sources

$$f_{x_j}(t, x) := \delta(t)\delta_{x_j}(x), \quad j = 1, 2, \dots$$

In order to study the measurements  $Lf_{x_j}$ , we will use Sobolev spaces, see [56],

$$\begin{aligned} H_p^s(\mathbb{R}^d) &:= \{f \in \mathcal{S}'(\mathbb{R}^d); \|f\|_{H_p^s(\mathbb{R}^d)} := \|(1 - \Delta)^{s/2} f\|_{L^p(\mathbb{R}^d)} < +\infty\}, \\ \tilde{H}_p^s(U) &:= \{f \in H_p^s(\mathbb{R}^d); \text{supp } f \subset \bar{U}\}, \\ H_p^s(U) &:= \{f \in \mathcal{D}'(U); f = h|_U \text{ for some } h \in H_p^s(\mathbb{R}^d)\}, \end{aligned}$$

where  $U \subset \mathbb{R}^d$  is open and  $s \in \mathbb{R}$ . When  $p = 2$  we omit the subscript  $p$  in our notation, that is, we denote  $H^s(U) = H_2^s(U)$  etc. Moreover, we use projective topology on the tensor product  $X \otimes Y$  of two Banach  $X$  and  $Y$ , that is,  $\|z\|_{X \otimes Y} := \inf \sum_j \|x_j\|_X \|y_j\|_Y$ , where infimum is taken over all representations  $z = \sum_j x_j \otimes y_j$ . We also use projective topology on tensor products of locally convex spaces, see e.g. [55, Def. 43.2]. The measurement  $Lf_{x_j}$  can be defined in the sense of the following lemma.

**Lemma 1.** *Let  $p \in (1, \frac{n}{n-1})$  and let  $m \in \mathbb{N}$  satisfy  $m > \frac{n+1}{4}$ . Then the measurement operator  $L$ , defined in (2) has a unique continuous extension*

$$L : \tilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \rightarrow \mathcal{D}'((T_0, T) \times \partial M).$$

We will prove Lemma 1 and other results presented in introduction in Sections 3-6.

In this paper we study a single measurement  $Lh_0$  that simultaneously combines all the measurements  $Lf_{x_j}$  by adding those together with appropriate weights. When the measurements  $Lf_{x_j}$  are summed together, to the authors knowledge, there are no algorithms which could filter the value of a particular measurement from the sum. We will ask, however, can we find

the essential features given by these measurements, like the travel times between points on  $\partial M$ , so that the metric could be determined under certain geometric conditions.

In Section 2, Definition 1, we construct a specific function  $h_0(t, x)$ , called *pseudorandom noise*, so that  $Lh_0$  determines the *scattering relation*  $\Sigma_{M,g}$  for the manifold  $(\overline{M}, g)$ . The scattering relation has been efficiently used to solve several geometric inverse problems [13, 45, 50, 51].

To define the scattering relation, let  $TM$  denote the tangent space of  $M$  and let  $\dot{\gamma}$  denote the tangent vector of a smooth curve  $\gamma : [a, b] \rightarrow M$ . Let  $SM = \{(x, \xi) \in TM; \|\xi\|_g = 1\}$  be the unit sphere bundle on  $M$  and define

$$\partial_{\pm} SM = \{(x, \xi) \in SM; x \in \partial M, \mp(\nu, \xi)_g > 0\},$$

where  $\nu$  is the exterior normal vector of  $\partial M$ . Moreover, let  $\tau(x, \xi)$  be the infimum of the set  $\{t \in (0, \infty]; \gamma_{x,\xi}(t) \in \partial M\}$ , where  $\gamma_{x,\xi}$  denotes the geodesic with initial data  $(x, \xi) \in TM$ . We define the infimum of empty set to be  $+\infty$ .

The scattering relation is the map  $\Sigma = \Sigma_{M,g}$ ,

$$\Sigma : \mathcal{D}(\Sigma) \rightarrow \overline{\partial_+ SM} \times \mathbb{R}, \quad \mathcal{D}(\Sigma) = \{(x, \xi) \in \partial_- SM; \tau(x, \xi) < \infty\}$$

defined by  $\Sigma(x, \xi) = (\gamma_{x,\xi}(\tau(x, \xi)), \dot{\gamma}_{x,\xi}(\tau(x, \xi)), \tau(x, \xi))$ .

Our main result is the following.

**Theorem 1.** *Let  $M \subset \mathbb{R}^n$ ,  $n \geq 2$  be an open and bounded set having a  $C^\infty$  smooth boundary. Then there is a generalized function  $h_0(t, x)$  such that it is supported on  $\{0\} \times \partial M$  and has the following properties: Assume that  $g_{jk}, g'_{jk} \in C^\infty(\mathbb{R}^n)$  are two Riemannian metric tensors satisfying (1). Moreover, assume that  $g_{jk}(x) = g'_{jk}(x)$  for  $x \in \mathbb{R}^n \setminus M$ . Let  $T > \max(\text{diam}(M, g), \text{diam}(M, g'))$ ,  $T_0 < 0$ , and assume that*

$$L_g h_0 = L_{g'} h_0 \quad \text{on } (T_0, T) \times \partial M.$$

*Then the scattering relations  $\Sigma_{M,g}$  and  $\Sigma_{M',g'}$  of Riemannian manifolds  $(M, g)$  and  $(M, g')$  are the same. In particular, if  $(\overline{M}, g)$  and  $(\overline{M}, g')$  are simple, the restrictions of the distance functions on the boundary satisfy  $d_{M,g}(x, y) = d_{M,g'}(x, y)$  for  $x, y \in \partial M$ .*

Recall that a compact Riemannian manifold  $(\overline{M}, g)$  with boundary is simple if it is simply connected, any geodesic has no conjugate points and

$\partial M$  is strictly convex with respect to the metric  $g$ . Any two points of a simple manifold can be joined by a unique geodesic.

The key idea of proof of Theorem 1 is to use source  $h_0(t, x) = \sum_{j=1}^{\infty} a_j f_{x_j}$ . The point source  $a_{j_0} f_{x_{j_0}}$  produces a singularity, which is observed at a point  $y \in \mathbb{R}^n \setminus M$  at time  $t_0 = d(x_{j_0}, y)$  with a magnitude  $a_{j_0} \beta(x_{j_0}, y)$ , where  $\beta$  is an unknown nonvanishing smooth function. Appropriate choice of the weights  $a_j$  allows us find the index  $j_0$  by looking nearby singularities. Indeed, when  $x_{j_k} \rightarrow x_{j_0}$  and  $j_k \rightarrow \infty$ , we see that the asymptotic behavior of the magnitude  $a_{j_k} \beta(x_{j_k}, y)$  as  $k \rightarrow \infty$  will be that of the weights  $a_{j_k}$ . Thus it is possible to factor out  $a_{j_k}$  in the magnitude and determine  $a_{j_0}$ . This argument is presented in Section 7 and gives us the distances  $d(x_j, y)$  in  $(\mathbb{R}^n, g)$  for arbitrary point  $y \in \mathbb{R}^n \setminus M$  and a source point  $x_j$ .

Theorem 1 and boundary rigidity results for simple manifolds imply the following:

**Corollary 1.** *Let  $M \subset \mathbb{R}^n$  and let  $g_{jk}, g'_{jk} \in C^\infty(\mathbb{R}^n)$  be two Riemannian metric tensors satisfying assumptions of Theorem 1. Assume that  $(\overline{M}, g)$  and  $(\overline{M}, g')$  are simple Riemannian manifolds. Then*

(i) *If  $n = 2$  and*

$$L_g h_0 = L_{g'} h_0 \quad \text{on } (T_0, T) \times \partial M \quad (4)$$

*then there is a diffeomorphism  $\Phi : M \rightarrow M$  such that  $\Phi|_{\partial M} = Id$  and  $g = \Phi_* g'$ .*

(ii) *For  $n \geq 3$  there is  $\epsilon = \epsilon_{n,M} > 0$  such that if  $\|g_{jk} - \delta_{jk}\|_{C^2(M)} < \epsilon_n$ ,  $\|g'_{jk} - \delta_{jk}\|_{C^2(M)} < \epsilon_n$  and (4) holds, then there is a diffeomorphism  $\Phi : M \rightarrow M$  such that  $\Phi|_{\partial M} = Id$  and  $g = \Phi_* g'$ .*

(iii) *If  $g_{jk}(x) = a(x)\delta_{jk}$  and  $g'_{jk}(x) = a'(x)\delta_{jk}$ , that is, the metric tensors are conformally Euclidian, and (4) holds, then  $g_{jk}(x) = g'_{jk}(x)$  for  $x \in M$ .*

Indeed, by Theorem 1, the case (i) follows from [44], (ii) follows from [10], and (iii) from [39, 40, 41].

If Uhlmann's conjecture [57], that the scattering relation determines the isometry type of non-trapping compact manifolds with non-empty boundary, can be proven, then Corollary 1 holds for more general class of manifolds.

The problem of determining the metric  $g$  (possibly up to a diffeomorphism) with given the measurement  $Lh_0$  with only one function  $h_0(t, x)$  is

a formally determined inverse problem. Indeed, the formally computed “dimension of the data”, that is the dimension of  $(T_0, T) \times \partial M$ , is  $n$  and coincides with dimension of the set  $M$  on which the unknown functions  $g_{jk}(x)$  are defined.

The formally determined inverse problems have been studied in many cases. For instance, two dimensional Calderon’s inverse problem [3, 4, 22, 43, 53] is formally determined. The same is true for the related inverse problem for the Schrödinger equation in two dimensions [9]. The corresponding inverse problems in dimension  $n \geq 3$ , see [11, 28, 34, 42, 54] and references in [16], are over-determined, that is, the dimension of the data is larger than the dimension of the unknown object. Similar classification holds for the elliptic inverse problems on Riemannian manifolds [17, 18, 34, 35, 37]. Moreover, the inverse travel time problems, i.e. boundary rigidity problem, see [30, 38, 39, 40, 41, 48, 52], is formally determined in dimension  $n = 2$  and overdetermined for  $n \geq 3$ .

Inverse problems in time domain related to the Laplace-Beltrami operator  $\Delta_g$ , namely the inverse boundary value problem for the wave, heat, and the dynamical Schrödinger equations with Dirichlet-to-Neumann as data, see [2, 7, 25, 26], are overdetermined in dimensions  $n \geq 2$ . However, these problems are equivalent to the inverse boundary spectral problem, see [27], and assuming that the eigenvalues are simple, Dirichlet-to-Neumann map at a generic Dirichlet boundary value determines the boundary spectral data [32, 33, 47]. Thus under generic conditions on the spectrum and on the boundary value (that is, under conditions that the these data belong in some open and dense set) it is possible to solve a formally determined inverse problem in time domain.

We point out that in this paper we do not impose any generic conditions on the geometry and we give an explicit construction of the boundary source. The boundary source considered in this paper is based on the idea of imitating a realization of white noise, and due to the many useful properties of white noise process, we hope that the constructed source may be useful in the study of other inverse problems requiring generic assumptions on the source.

Another formally determined hyperbolic inverse problem, namely measuring Neumann data when the initial data  $(u|_{t=0}, \partial_t u|_{t=0})$  is non-zero and satisfies subharmonicity or positivity conditions, has been studied using Carleman estimates [8, 23, 29]. The present paper is closely related to these studies, but we emphasize that we assume that the initial data for  $u$  vanishes.

## 2 Pseudorandom noise as a source

In this section we define a special source  $h_0(t, x)$  which we call the *pseudorandom noise*. The specific assumptions on the amplitudes are explained in Section 7. An important feature of the pseudorandom noise is that it is supported only on a single point in time.

**Definition 1.** Let  $x_j \in \partial M$ ,  $j = 1, 2, \dots$ , be a dense sequence of disjoint points in  $\partial M$ , and let  $a_j \in \mathbb{R}$ ,  $j = 1, 2, \dots$ ,  $\sum_{j=1}^{\infty} |a_j| < \infty$  be a sequence of disjoint numbers.

We define the pseudorandom noise on  $(x_j)_{j=1}^{\infty} \subset \partial M$  with coefficients  $(a_j)_{j=1}^{\infty} \subset \mathbb{R}$  as the following generalized function on  $\mathbb{R} \times \mathbb{R}^n$ :

$$h_0(t, x) := \sum_{j=1}^{\infty} a_j \delta(t) \delta_{x_j}(x), \quad (x, t) \in \mathbb{R}^{n+1},$$

where  $\delta(t)$  and  $\delta_{x_j}(x)$  are Dirac delta distributions on  $\mathbb{R}$  and  $\mathbb{R}^n$ , respectively.

It is rather straightforward to show that  $h_0$  is well-defined. First, it is well known that  $\delta(t) \in H^{-1}(\mathbb{R})$  and  $\delta_{x_j}(x) \in C(\mathbb{R}^n)'$ . Moreover,  $H_{p'}^1(\mathbb{R}^n) \subset C(\mathbb{R}^n)$  when  $1 > n/p'$  due to [56, Thm. 2.8.1]. According to [56, Thm. 2.6.1] the dual space satisfies  $(H_{p'}^1(\mathbb{R}^n))' = H_p^{-1}(\mathbb{R}^n)$  with  $1/p' = 1 - 1/p$  and hence  $C(\mathbb{R}^n)' \subset H_p^{-1}(\mathbb{R}^n)$  for  $1 < p < \frac{n}{n-1}$ . Since  $\sum_{j=1}^{\infty} |a_j| < \infty$  we have

$$\sum_{j=1}^{\infty} a_j \delta_{x_j}(x) \in H_p^{-1}(\mathbb{R}^n).$$

This yields that for any  $p \in (1, \frac{n}{n-1})$  and  $\epsilon > 0$  the pseudorandom noise  $h_0$  satisfies

$$h_0 \in \tilde{H}^{-1}(-\epsilon, \epsilon) \otimes \tilde{H}_p^{-1}(M). \quad (5)$$

The spatial structure of the pseudorandom noise can be motivated by the structure the white noise. In the 1-dimensional radar imaging models, white noise signals are considered to be optimal sources when imaging a stationary scatterer [19]. This is due to the fact that different translations of the white noise signal are uncorrelated. In a similar fashion we have the following property for the pseudorandom noise  $h_0$ : for each  $x_{j_0}$  and each sequence  $(x_{j_k})_{k=1}^{\infty}$  converging to  $x_{j_0}$  and satisfying  $x_{j_k} \neq x_{j_0}$  for all  $k \in \mathbb{Z}_+$ , it holds that  $a_{j_k} \rightarrow 0$ . This property will be crucial in what follows.

Moreover, a natural strategy to choose the points  $x_j$  is by random sampling. The term pseudorandom refers to the fact that the algorithmic generators of random numbers use, in fact, a deterministic function to produce a sequences of numbers through so mixing process, that the user of the algorithm can consider the numbers to be analogous to independent samples of a random variable. In this manner, the pseudorandom noise can be seen as an imitation of a realization of a noise process.

Another source of inspiration for us was a rather new measurement paradigm called compressed sensing [12, 14], where one aims for a sparse reconstructions of a linear problem using a small number of noisy measurements. We point out that by using the pseudorandom noise one can compress the measurements  $Lf_{x_j}$  with point sources  $f_{x_j}$  into a single measurement  $Lh_0$ .

### 3 Measurement map

In this section we prove that the measurement is well-defined when we have pseudorandom noise as source.

Next, we consider the operator  $W : f \mapsto u$  mapping  $f$  to the solution of the equation (3). We call such operator the solution operator for the equation (3). First, we note that by [21, Thm. 23.2.2], the operator  $W : f \mapsto u$  mapping  $f$  extends in a unique way to a continuous linear operator

$$W : L^1((T_0, T); H^s(\mathbb{R}^n)) \rightarrow C([T_0, T]; H^{s+1}(\mathbb{R}^n)), \quad s \in \mathbb{R}. \quad (6)$$

Moreover, if  $f \in C^\infty([T_0, T] \times \mathbb{R}^n)$  and  $\text{supp}(f) \subset\subset (T_0, T] \times \mathbb{R}^n$ , that is,  $\text{supp}(f)$  is a compact subset of  $(T_0, T] \times \mathbb{R}^n$ , then  $Wf \in C^\infty([T_0, T] \times \mathbb{R}^n)$ .

We will compose the operator  $W$  with the one-sided inverse  $\mathcal{I}$  of the derivative  $\partial_t$ , which is given by

$$\mathcal{I}u(t) := \int_{T_0}^t u(t') dt', \quad u \in C_c^\infty(T_0, T).$$

One sees easily that this operator has a unique continuous linear extension  $\mathcal{I} : \tilde{H}^{-1}(T_0, T) \rightarrow L^2(T_0, T)$ .

Next we prove that the measurement map  $L$  has unique continuous extension

$$\tilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \rightarrow \mathcal{D}'((T_0, T) \times \partial M). \quad (7)$$



*Proof of Lemma 1.* For sufficiently large  $z \in \mathbb{R}_+$ , operator  $z - \Delta_g$  is an isomorphism between spaces  $H^{s+2}(\mathbb{R}^n)$  and  $H^s(\mathbb{R}^n)$  as well as between spaces  $H_p^{s+2}(\mathbb{R}^n)$  and  $H_p^s(\mathbb{R}^n)$  for all integers  $s$  by [49].

By the definition of  $L$ , we have that  $L = \text{Tr} \circ W$ , where  $\text{Tr}$  is the trace operator

$$\text{Tr}(u) = u|_{(T_0, T) \times \partial M}, \quad u \in C^\infty((T_0, T) \times \mathbb{R}^n).$$

Let  $f \in C_c^\infty((T_0, T) \times \mathbb{R}^n)$ . Then the solution  $u = Wf$  of the wave equation  $(\partial_t^2 - \Delta_g)u = f$  can be written in the form

$$Wf = (z - \partial_t^2)^m (z - \Delta_g)^{-m} Wf + \sum_{j=0}^{m-1} (z - \partial_t^2)^j (z - \Delta_g)^{-1-j} f. \quad (8)$$

Now  $f = \partial_t \mathcal{I}f$ , where  $\mathcal{I}f$  is  $C^\infty$ -smooth and satisfies  $\text{supp}(\mathcal{I}f) \subset\subset (T_0, T] \times \mathbb{R}^n$ . By (6),  $W\mathcal{I}f$  is  $C^\infty$ -smooth and  $\partial_t W\mathcal{I}f = W\partial_t \mathcal{I}f = Wf$ . Hence

$$Lf = \partial_t (z - \partial_t^2)^m \text{Tr}(z - \Delta_g)^{-m} W\mathcal{I}f + \sum_{j=0}^{m-1} (z - \partial_t^2)^j \text{Tr}(z - \Delta_g)^{-1-j} f. \quad (9)$$

Let us next consider terms appearing in (9). First we consider extension of the operator

$$\begin{aligned} \sum_{k=1}^N \phi_k \otimes \psi_k &\mapsto \sum_{k=1}^N (z - \partial_t^2)^j \text{Tr}(z - \Delta_g)^{-1-j} (\phi_k \otimes \psi_k) \\ &= \sum_{k=1}^N ((z - \partial_t^2)^j \phi_k) \otimes (\text{Tr}(z - \Delta_g)^{-1-j} \psi_k), \quad j = 0, \dots, m-1, \end{aligned} \quad (10)$$

mapping  $C_c^\infty(T_0, T) \otimes C_c^\infty(\mathbb{R}^n)$  to  $C^\infty((T_0, T) \times \partial M)$ . By [56, Thm 4.7.1] the maps

$$H_p^{-1}(\mathbb{R}^n) \xrightarrow{(z - \Delta_g)^{-1-j}} H_p^1(\mathbb{R}^n) \xrightarrow{\text{Tr}} B_{p,p}^{1-1/p}(\partial M)$$

are continuous, where  $B_{p,p}^{1-1/p}(\partial M)$  is the Besov space on  $\partial M$ . Thus the operator (10) has a continuous extension in spaces (7).

Next, consider the extensions of the operator

$$\sum_{k=1}^N \phi_k \otimes \psi_k \mapsto \sum_{k=1}^N \partial_t (z - \partial_t^2)^m \text{Tr}(z - \Delta_g)^{-m} W((\mathcal{I}\phi_k) \otimes \psi_k) \quad (11)$$

mapping  $C_c^\infty(T_0, T) \otimes C_c^\infty(\mathbb{R}^n)$  to  $C^\infty((T_0, T) \times \partial M)$ . As  $-1-n/p > -1-n$  we have by [56, Thm. 2.8.1] a continuous embedding  $H_p^{-1}(\mathbb{R}^n) \hookrightarrow H^{-1-n/2}(\mathbb{R}^n)$ . Moreover, the operator  $\mathcal{I} : \tilde{H}^{-1}(T_0, T) \rightarrow L^2(T_0, T)$  and the embedding  $L^2(T_0, T) \otimes H^{-1-n/2}(\mathbb{R}^n) \hookrightarrow L^2((T_0, T); H^{-1-n/2}(\mathbb{R}^n))$  are continuous. Thus, by (6),

$$W\mathcal{I} : \tilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \rightarrow C([T_0, T]; H^{-n/2}(\mathbb{R}^n))$$

is continuous.

Thus, as  $(1 - \Delta_g)^{-m} : C([T_0, T]; H^{-n/2}(\mathbb{R}^n)) \rightarrow C([T_0, T]; H^{-n/2+2m}(\mathbb{R}^n))$  is continuous and  $-n/2 + 2m > 1/2$ , we see that the operator

$$\text{Tr}(1 - \Delta_g)^{-m} W\mathcal{I} : \tilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \rightarrow C([T_0, T]; L^2(\partial M))$$

is continuous.

Combining the above results, we see that the operator (9) has a continuous extension to spaces (7). As the spaces  $C_c^\infty(T_0, T)$  and  $C_c^\infty(\mathbb{R}^n)$  are dense in  $\tilde{H}^{-1}(T_0, T)$  and  $H_p^{-1}(\mathbb{R}^n)$ , respectively, we see that the continuous extension of  $L$  is unique.  $\square$

## 4 Inner product of a solution and a source

**Lemma 2.** *Let  $f \in C_c^\infty((T_0, T) \times M)$ ,  $t_0 \in (T_0, T)$  and let  $w \in C^\infty([T_0, t_0] \times \mathbb{R}^n)$  satisfy*

$$(\partial_t^2 - \Delta_g)w = 0, \quad \text{in } (T_0, t_0) \times \mathbb{R}^n.$$

*Then*

$$\begin{aligned} & \int_{T_0}^{t_0} \int_{\mathbb{R}^n} f(t, x) w(t, x) dt dV(x) \\ &= \int_{\mathbb{R}^n} ((\partial_t W f)(t_0, x) w(t_0, x) - (W f)(t_0, x) (\partial_t w)(t_0, x)) dV(x) \end{aligned}$$

*where  $dV(x) = |g|^{1/2} dx$  is the Riemannian volume measure of  $(\mathbb{R}^n, g)$  and  $W : f \mapsto u$  is the solution operator of wave equation (3).*

*Proof.* By finite speed of propagation of waves, see e.g. [31, pp. 150-156],  $\text{supp}(Wf(t))$  is compact in  $\mathbb{R}^n$ . The claim follows by integration by parts

$$\begin{aligned}
& \int_{\mathbb{R}^n} ((\partial_t u)(t_0, x)w(t_0, x) - u(t_0, x)(\partial_t w)(t_0, x))dV(x) \\
& - \int_{\mathbb{R}^n} ((\partial_t u)(T_0, x)w(T_0, x) - u(T_0, x)(\partial_t w)(T_0, x))dV(x) \\
& = \int_{(T_0, t_0) \times \mathbb{R}^n} ((\partial_t^2 - \Delta_g)u(t, x)w(t, x) - u(t, x)(\partial_t^2 - \Delta_g)w(t, x))dtdV(x) \\
& = \int_{(T_0, t_0) \times \mathbb{R}^n} f(t, x)w(t, x)dtdV(x).
\end{aligned}$$

□

Next, we will prove a generalization of the previous lemma for non-smooth sources  $f$ . Denote by  $B(0, R) = \{x \in \mathbb{R}^n; |x| < R\}$  the Euclidean ball. The finite speed of propagation for wave equation, yields that there is  $R > 0$  such that all  $f \in C_c^\infty((T_0, T) \times M)$  satisfy  $\text{supp}(Wf) \subset\subset (T_0, T] \times B(0, R)$ . We define

$$\Omega := B(0, R) \setminus \overline{M}. \quad (12)$$

Below, we use the fact (see [15, Thm. 7.2.3/6, Thm. 5.6.3/6]) that the operator  $W_\Omega : h \mapsto v$  mapping  $h$  to the solution of the equation

$$\begin{aligned}
& (\partial_t^2 - \Delta_g)v(t, x) = 0 \quad \text{in } (T_0, T) \times \Omega, \\
& v|_{(T_0, T) \times \partial\Omega} = h, \\
& v|_{t=T_0} = 0, \quad \partial_t v|_{t=T_0} = 0.
\end{aligned} \quad (13)$$

is continuous  $W_\Omega : C_c^\infty((T_0, T) \times \partial\Omega) \rightarrow C^\infty([T_0, T] \times \overline{\Omega})$ .

We let  $t_0 \in (T_0, T)$  and denote

$$\Sigma := \{t_0\} \times \Omega \quad (14)$$

We denote the trace on  $\Sigma$  by  $\text{Tr}_\Sigma$ , that is, we define  $(\text{Tr}_\Sigma u)(x) := u(t_0, x)$ . Let  $\nu = \nu(z)$  denote the exterior unit normal vector of  $\partial M$  at  $z$ .

Moreover, let  $U$  be an open subset (or a submanifold) of  $\mathbb{R}^n$  and let us denote by  $dV$  (or  $dS$ ) the Riemannian volume measure of  $(U, g)$ . We embed the test functions into the spaces of distribution by using the inner product of the space  $L^2(U; dV)$ , that is, we identify  $u \in C_0^\infty(U)$  with the distribution

$$\psi \mapsto \int_U u(x)\psi(x) dV(x). \quad (15)$$

We will denote the distribution pairing of  $u \in \mathcal{D}'(U)$  and  $\psi \in C_0^\infty(U)$  by  $(u, \psi)_{\mathcal{D}'(U)}$  and use analogous notations for other distribution pairings.

**Lemma 3.** *Let  $t_0 \in (T_0, T)$  and define  $\Sigma$  by (14). Then operators  $\text{Tr}_\Sigma W_\Omega$  and  $\text{Tr}_\Sigma \partial_t W_\Omega$  have unique continuous extensions  $\mathcal{E}'((T_0, t_0) \times \partial\Omega) \rightarrow \mathcal{D}'(\Omega)$ .*

*Proof.* Let  $v$  satisfy (13). Consider a function  $w \in C^\infty([T_0, t_0] \times \overline{\Omega})$  such that  $(\partial_t - \Delta_g)w = 0$  in  $(T_0, t_0) \times \Omega$  and  $w|_{(T_0, t_0) \times \partial\Omega} = 0$ . Then

$$\begin{aligned} 0 &= \int_{\Omega \times (T_0, t_0)} ((\partial_t - \Delta_g)v)w - v((\partial_t - \Delta_g)w) dV(x) dt \\ &= \left[ \int_{\Omega} ((\partial_t v)w - v(\partial_t w)) dV(x) \right]_{t=T_0}^{t=t_0} + \int_{\partial\Omega \times (T_0, t_0)} ((\partial_\nu v)w - v(\partial_\nu w)) dS(x) dt \\ &= \int_{\Omega} ((\partial_t v)w - v(\partial_t w)) dV(x) \Big|_{t=t_0} - \int_{\partial\Omega \times (T_0, t_0)} h(\partial_\nu w) dS(x) dt, \end{aligned}$$

where  $\partial_\nu$  is the normal derivative on  $\partial\Omega$ ,

Denote by  $W_1 : f_1 \mapsto w$  the solution operator of the equation

$$\begin{aligned} (\partial_t - \Delta_g)w(t, x) &= 0 \quad \text{in } (T_0, t_0) \times \Omega, \\ w|_{(T_0, t_0) \times \partial\Omega} &= 0, \\ w|_{t=t_0} &= f_1, \quad \partial_t w|_{t=t_0} = 0. \end{aligned}$$

Operator  $W_1 : C_c^\infty(\Omega) \rightarrow C^\infty([T_0, t_0] \times \overline{\Omega})$ , is continuous, as can be seen using [15, Thm. 7.2.3/6, Thm. 5.6.3/6]. Hence the operator

$$\partial_\nu W_1 : C_c^\infty(\Omega) \rightarrow C^\infty([T_0, t_0] \times \partial\Omega), \quad f \mapsto \partial_\nu W_1 f|_{\partial\Omega}$$

is continuous. Moreover,

$$(\text{Tr}_\Sigma \partial_t W_\Omega h, f_1)_{L^2(\Omega; dV)} = (h, \partial_\nu W_1 f_1)_{L^2((T_0, t_0) \times \partial\Omega; dt \otimes dS)},$$

where  $\partial_\nu$  is the normal derivative on  $\partial\Omega$ . We define the extension of  $\text{Tr}_\Sigma \partial_t W_\Omega$  by identifying it with the transpose  $(\partial_\nu W_1)^t : \mathcal{E}'((T_0, t_0) \times \partial\Omega) \rightarrow \mathcal{D}'(\Omega)$  of the operator  $\partial_\nu W_1 : C_c^\infty(\Omega) \rightarrow C^\infty([T_0, t_0] \times \partial\Omega)$ .

Similarly, we define the extension of  $\text{Tr}_\Sigma W_\Omega$  by the transpose  $(\partial_\nu W_2)^t : \mathcal{E}'((T_0, t_0) \times \partial\Omega) \rightarrow \mathcal{D}'(\Omega)$  of  $\partial_\nu W_2 : C_c^\infty(\Omega) \rightarrow C^\infty([T_0, t_0] \times \partial\Omega)$ , where  $W_2 : f_2 \mapsto w$  is the solution operator of the equation

$$\begin{aligned} (\partial_t - \Delta_g)w(t, x) &= 0 \quad \text{in } (T_0, t_0) \times \Omega, \\ w|_{(T_0, t_0) \times \partial\Omega} &= 0, \\ w|_{t=t_0} &= 0, \quad \partial_t w|_{t=t_0} = -f_2. \end{aligned}$$

□

Denote by  $d_\Omega(x, y)$ ,  $x, y \in \overline{\Omega}$ , the distance function of Riemannian manifold  $(\overline{\Omega}, g|_{\overline{\Omega}})$ . Next we generalize the result of Lemma 2 for a larger class of functions.

**Lemma 4.** *Let  $t_0 \in (0, T)$  and  $\epsilon > 0$  satisfy  $[-\epsilon, \epsilon] \subset (T_0, t_0)$ . Define  $\Sigma$  by (14). Let  $f \in \tilde{H}^{-1}(-\epsilon, \epsilon) \otimes \tilde{H}_p^{-1}(M)$  and  $w \in C^\infty([T_0, t_0] \times \mathbb{R}^n)$  satisfy*

$$(\partial_t^2 - \Delta_g)w = 0, \quad \text{in } (T_0, t_0) \times \mathbb{R}^n.$$

*Suppose that  $w(t_0), \partial_t w(t_0) \in C_c^\infty(\Omega)$ , and let  $\chi \in C_c^\infty(T_0, t_0)$  satisfy  $\chi = 1$  in a neighborhood of  $[-\epsilon, t_0 - r]$ , where*

$$r := d_\Omega(\text{supp}(w(t_0)) \cup \text{supp}(\partial_t w(t_0)), \partial\Omega).$$

*Then*

$$\begin{aligned} (f, w)_{\mathcal{E}'(\mathbb{R}^n \times (T_0, t_0))} \\ = (\text{Tr}_\Sigma \partial_t W_\Omega \chi Lf, w)_{\mathcal{D}'(\Omega)} - (\text{Tr}_\Sigma W_\Omega \chi Lf, \partial_t w)_{\mathcal{D}'(\Omega)}, \end{aligned} \quad (16)$$

*where we have defined  $Lf = 0$  on  $\partial B(0, R)$ . Here we regard  $\Omega$  as Riemannian manifold  $(\Omega, g|_\Omega)$ .*

*Proof.* We suppose first that  $f \in C_c^\infty((-\epsilon, \epsilon) \times M)$ . Recall that  $W$  is solution operator of wave equation (3). Then  $Wf(\cdot, t) = 0$  if  $t < -\epsilon$ , and

$$Lf = \text{Tr}_{\partial\Omega} Wf = \chi \text{Tr}_{\partial\Omega} Wf, \quad \text{in } (T_0, t_0 - r) \times \partial\Omega,$$

where  $\text{Tr}_{\partial\Omega}$  is the trace on  $(T_0, T) \times \partial\Omega$ . As  $\Omega \cap \overline{M} = \emptyset$ , we have that  $(\partial_t^2 - \Delta_g)Wf = 0$  in  $(T_0, T) \times \Omega$ . By uniqueness of the solution of (13)

$$W_\Omega \chi \text{Tr}_{\partial\Omega} Wf = Wf, \quad \text{in } (T_0, t_0 - r) \times \Omega.$$

By finite speed of propagation

$$\text{Tr}_\Sigma \partial_t^j W_\Omega \chi \text{Tr}_{\partial\Omega} Wf = \text{Tr}_\Sigma \partial_t^j Wf, \quad j = 0, 1,$$

on  $\{t_0\} \times \text{supp}(w(t_0)) \cup \text{supp}(\partial_t w(t_0))$ . By Lemma 2, (16) holds.

Then the claim follows as the embeddings

$$C_c^\infty(-\epsilon, \epsilon) \hookrightarrow \tilde{H}^{-1}(-\epsilon, \epsilon), \quad C_c^\infty(M) \hookrightarrow \tilde{H}_p^{-1}(M)$$

are dense and operators  $(\text{Tr}_\Sigma \partial_t^j W_\Omega) \chi L : \tilde{H}^{-1}(-\epsilon, \epsilon) \otimes \tilde{H}_p^{-1}(M) \rightarrow \mathcal{D}'((T_0, t_0) \times \partial\Omega)$ ,  $j = 0, 1$ , are continuous.  $\square$

## 5 Gaussian beams

We consider solutions of wave equation which are known as Gaussian beams [5, 6, 46]. These solutions have been constructed to analyze the propagation of singularities for the wave equation in the presence of caustics. Here we use Gaussian beams as an auxiliary technical tool to analyze singularities in the measurements.

**Definition 2.** *Let  $\epsilon > 0$ ,  $N \in \mathbb{N}$  and let  $\gamma$  be a unit speed geodesic on  $(\mathbb{R}^n, g)$ . A formal Gaussian beam of order  $N$  propagating along geodesic  $\gamma$  is a function  $U_\epsilon^N$  of form*

$$U_\epsilon^N(t, x) = \epsilon^{-n/4} \exp \{-(i\epsilon)^{-1} \theta(t, x)\} \sum_{m=0}^N u_m(t, x) (i\epsilon)^m, \quad t \in \mathbb{R}, x \in \mathbb{R}^n$$

*satisfying the following properties: The phase function  $\theta$  and the amplitude functions  $u_m$ ,  $m = 0, 1, \dots, N$ , are complex valued smooth functions. The phase function  $\theta$  satisfies the conditions*

$$\theta(t, \gamma(t)) = 0, \quad \text{Im } \theta(t, x) \geq C_0(t) d(x, \gamma(t))^2$$

*where  $C_0(t)$  is a continuous strictly positive function. The amplitude function  $u_0$  satisfies  $u_0(t, \gamma(t)) \neq 0$ . Finally, for any compact set  $K \subset \subset \mathbb{R} \times \mathbb{R}^n$  there is a constant  $C > 0$  such that the inequality*

$$|(\partial_t^2 - \Delta_g) U_\epsilon^N(t, x)| \leq C \epsilon^{N-n/4}$$

*is satisfied uniformly for  $(t, x) \in K$ .*

The construction of a formal Gaussian beam  $U_\epsilon^N(t, x)$  is considered in detail e.g. in [26, Sect. 2.4]. Next, we recall the construction and pay attention to the properties of Gaussian beams which we need later.

Let us write the geodesic  $\gamma$  in the usual coordinates of  $\mathbb{R}^n$  as  $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$ . We construct the phase function  $\theta(t, x)$  at each time  $t \in \mathbb{R}$  in terms of a finite Taylor expansion in the  $x$  variable centered at  $\gamma(t)$ ,

$$\theta(t, x) = \sum_{|\alpha| \leq N} \frac{\theta_\alpha(t)}{\alpha!} (x - \gamma(t))^\alpha,$$

where  $\theta_\alpha$  are smooth functions and  $N \in \mathbb{N}$ .

Let  $e_j = (\delta_{1j}, \dots, \delta_{nj})$  be multi-indexes with the value 1 at the  $j$ th place. For clarity, we use the notation  $p_j(t) = \theta_{e_j}(t)$  for the first order coefficients and the notation  $H_{jk}(t) = \theta_{\alpha}(t)$ ,  $\alpha = e_j + e_k$ , for the second order coefficients in the expansion of  $\theta$ .

The construction of a formal Gaussian beam consists of the following steps.

1. We define  $\theta_0(t) = 0$  and  $p_j(t) = \sum_{k=1}^n g_{jk}(\gamma(t)) \dot{\gamma}^k(t)$ , that is, the first order coefficients  $p_j(t)$  are the covariant representation of the velocity vector  $\dot{\gamma}$ .
2. The symmetric matrix  $H(t) = [H_{jk}(t)]_{j,k=1}^n$  of the second order coefficients are obtained by solving a Riccati equation, or an equivalent system of ordinary differential equation. We write  $H(t) = Z(t)Y(t)^{-1}$ , where the pair of complex  $n \times n$  matrices  $(Z(t), Y(t))$  is the solution of the system of ordinary differential equations,

$$\begin{aligned} \frac{d}{dt}Y(t) &= B(t)^*Y(t) + C(t)Z(t), \quad Y|_{t=0} = Y^0, \\ \frac{d}{dt}Z(t) &= -D(t)Y(t) - B(t)Z(t), \quad Z|_{t=0} = Z^0. \end{aligned}$$

Here we choose the initial values to be  $Z^0 = iI$  and  $Y^0 = I$ , where  $I$  is the identity matrix and  $i$  is the imaginary unit. The matrices  $B(t)$ ,  $C(t)$ , and  $D(t)$  in  $\mathbb{R}^{n \times n}$  have components given by the second derivatives of the Hamiltonian  $h(x, p) = (\sum_{j,k=1}^n g_{jk}(x)p^j p^k)^{1/2}$  evaluated in the point  $(x, p) = (\gamma(t), p(t))$

$$B_l^j = \frac{\partial^2 h}{\partial x^l \partial p_j}; \quad C^{jl} = \frac{\partial^2 h}{\partial p_j \partial p_l}; \quad D_{jl} = \frac{\partial^2 h}{\partial x^j \partial x^l}.$$

The fact that the complex matrix  $Y(t)$  is invertible for all  $t \in \mathbb{R}$  is crucial for the construction, and is discussed in detail in [26, Section 2.4].

3. The coefficients  $\theta_{\alpha}(t)$  of order  $|\alpha| = m \geq 3$  are solved inductively, with respect to  $m$ . The coefficients  $\theta_{\alpha}(t)$  are constructed using the coefficients  $\tilde{\theta}_{\alpha}(t)$  defined so that

$$\sum_{|\alpha|=m} \tilde{\theta}_{\alpha}(t) \tilde{y}^{\alpha} = \sum_{|\alpha|=m} \theta_{\alpha}(t) (x - \gamma(t))^{\alpha},$$

for all  $\tilde{y} = Y^{-1}(t)(x - \gamma(t))$ ,  $y \in \mathbb{C}^n$ . We obtain the coefficients  $\tilde{\theta}_\alpha(t)$  by solving successive linear systems of ordinary differential equations

$$\frac{d}{dt}\tilde{\theta}_\alpha(t) = K_\alpha(t), \quad \tilde{\theta}_\alpha(0) = 0$$

where  $K_\alpha(t)$ , depend on  $\theta_\beta(t)$  with  $|\beta| \leq m-1$ , the matrix  $H(t)$ , vector  $p(t)$ , and the metric  $g_{jk}$  and its derivatives at  $\gamma(t)$ .

4. When the phase function  $\theta(t, x)$  is constructed, the amplitude functions  $u_n(t, x)$  are solved using the transport equations, or equivalently, the following ordinary differential equations. Let

$$u_m(t, x) = \sum_{|\alpha| \leq N} \tilde{u}_{m,\alpha}(t) \tilde{y}^\alpha, \quad \tilde{y} = Y^{-1}(t)(x - \gamma(t))$$

where the coefficients  $\tilde{u}_{m,\alpha}(t)$  are obtained by solving the successive equations

$$\frac{d}{dt}\tilde{u}_{m,\alpha}(t) + r(t)\tilde{u}_{m,\alpha}(t) = \mathcal{F}_{m,\alpha}(t), \quad \tilde{u}_{m,\alpha}(0) = \delta_{m,0}\delta_{|\alpha|,0},$$

where  $r(t)$  and  $\mathcal{F}_{m,\alpha}(t)$  depend on  $\tilde{u}_{m',\beta}$  with  $|\beta| \leq |\alpha| + 2$  and  $m' \leq m-1$ , the function  $\theta(t, x)$ , the metric  $g_{jk}$  and their derivatives at  $(t, x)$ ,  $x = \gamma(t)$ .

By the above construction, we have the following remark.

**Remark 1.** *The phase function  $\theta(t, x)$  and the amplitude functions  $u_m(t, x)$  at time  $t = 0$  have the form*

$$\theta(0, x) = \sum_{j,k=1}^n g_{jk}(y) \eta^k (x^j - y^j) + i|x - y|^2,$$

where  $(y, \eta) = (\gamma(0), \dot{\gamma}(0))$  is the initial data of the geodesic  $\gamma$ ,  $u_0(0, x) = 1$ , and  $u_m(0, x) = 0$  for  $m > 0$ . Hence  $U_\epsilon^N(0, x)$  is dependent on the metric  $g_{jk}$  only via  $g_{jk}(y)$ . Moreover,  $\partial_t U_\epsilon^N(0, x)$ , although of more complex form, is dependent on the metric  $g_{jk}$  only via  $\partial^\alpha g_{jk}(y)$  for a certain finite collection of multi-indices  $\alpha \in \mathbb{N}^n$ .



If the coefficients of an ordinary differential equation depend smoothly on some parameter so does the solution [1], and thus we see using an induction that the phase function  $\theta$  and the amplitude functions  $u_m$  depend smoothly on the initial data  $(y, \eta) = (\gamma(0), \dot{\gamma}(0))$  of the geodesic  $\gamma$ . In particular, the amplitude function  $u_0(t, x; y, \eta)$  satisfies

$$u_0 \in C^\infty(\mathbb{R} \times \mathbb{R}^n \times S\mathbb{R}^n). \quad (17)$$

To this far we have considered a formal Gaussian beam. By using continuous dependency of the solution of the wave equation on the source term, one obtains the following results, see e.g. [26]:

Let  $\gamma$  be a unit speed geodesic,  $N \in \mathbb{N}$ ,  $\epsilon > 0$  and let  $U_\epsilon^N$  be a formal Gaussian beam of order  $N$  propagating along geodesic  $\gamma$ . Let  $\chi \in C_0^\infty(\mathbb{R}^n)$  be a function which is identically one in a neighborhood of  $\gamma(0)$  and let  $t_0 > 0$  and let  $R$  be the radius in the equation (12). Then for  $j \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^n$  satisfying  $j + |\alpha| < N - n/4$  there is  $C > 0$  such that the solution  $w_\epsilon$  of the wave equation

$$\begin{aligned} (\partial_t^2 - \Delta_g)w_\epsilon(t, x) &= 0, \quad (t, x) \in (T_0, t_0) \times \mathbb{R}^n, \\ w_\epsilon(t_0, x) &= \chi(x)U_\epsilon^N(0, x), \quad \partial_t w_\epsilon(t_0, x) = -\chi(x)\partial_t U_\epsilon^N(0, x). \end{aligned} \quad (18)$$

satisfies

$$\sup_{x \in B(0, R), t \in (T_0, t_0)} |\partial_t^j \partial_x^\alpha (w_\epsilon(t_0 - t, x) - U_\epsilon^N(t, x))| \leq C\epsilon^{N-(j+|\alpha|)-n/4}. \quad (19)$$

We call  $w_\epsilon$  a Gaussian beam of order  $N$  propagating along geodesic  $\gamma$  backwards on time interval  $(T_0, t_0)$ .

## 6 Determination of the travel times

**Lemma 5.** *Let  $w_\epsilon$  be a Gaussian beam of order  $N \geq 1 + n/4$  propagating along geodesic  $\gamma$  backwards on time interval  $(T_0, t_0)$ , that is, let  $w_\epsilon$  be the solution of (18). Let  $h_0$  be the pseudorandom noise*

$$h_0(t, x) = \sum_{j=1}^{\infty} a_j \delta(t) \delta_{x_j}(x). \quad (20)$$

*If  $\gamma(t_0) \neq x_j$  for all  $j = 1, 2, \dots$  then*

$$\lim_{\epsilon \rightarrow 0} \epsilon^{n/4} (h_0, w_\epsilon)_{\mathcal{E}'(\mathbb{R}^n \times (T_0, t_0))} = 0.$$

Moreover, if  $\gamma(t_0) = x_j$  then

$$\lim_{\epsilon \rightarrow 0} \epsilon^{n/4} (h_0, w_\epsilon)_{\mathcal{E}'(\mathbb{R}^n \times (T_0, t_0))} = a_j u_0(t_0, x_j) |g|^{1/2}(x_j),$$

where  $u_0(t, x)$  is the first amplitude function of a formal Gaussian beam propagating along geodesic  $\gamma$ .

We remind the reader that the test functions are embedded in  $\mathcal{E}'(\mathbb{R}^n \times (T_0, T))$  using (15).

*Proof.* By equation (19) we have that

$$\begin{aligned} & \epsilon^{n/4} (h_0, w_\epsilon)_{\mathcal{E}'(\mathbb{R}^n \times (T_0, t_0))} \\ &= \epsilon^{n/4} \sum_{j=1}^{\infty} a_j U_\epsilon^N(t_0, x_j) |g|^{1/2}(x_j) + O(\epsilon) \\ &= \sum_{j=1}^{\infty} a_j u_0(t_0, x_j) \exp \{-(i\epsilon)^{-1} \theta(t_0, x_j)\} |g|^{1/2}(x_j) + O(\epsilon). \end{aligned}$$

As  $\text{Im } \theta(t_0, x_j) \geq C_0(t_0) d(x_j, \gamma(t_0))$  we have that

$$|\exp \{-(i\epsilon)^{-1} \theta(t_0, x_j)\}| = O(\epsilon), \quad \text{if } \gamma(t_0) \neq x_j.$$

Suppose that  $\gamma(t_0) = x_j$ . Then  $\exp \{-(i\epsilon)^{-1} \theta(t_0, x_j)\} = 1$  and there is a constant  $C > 0$  depending on  $\gamma$  and  $t_0$  such that

$$\begin{aligned} & |\epsilon^{n/4} (h_0, w_\epsilon)_{\mathcal{E}'(\mathbb{R}^n \times (T_0, t_0))} - a_j u_0(t_0, x_j) |g|^{1/2}(x_j)| \\ & \leq C \sum_{k=1}^{j-1} |a_k| |\exp \{-(i\epsilon)^{-1} \theta(t_0, x_k)\}| + C \sum_{k=j+1}^l |a_k| |\exp \{-(i\epsilon)^{-1} \theta(t_0, x_k)\}| \\ & \quad + C \sum_{l+1}^{\infty} |a_l| + O(\epsilon). \end{aligned}$$

We may first choose large  $l \in \mathbb{N}$  and then small  $\epsilon > 0$  so that the above three sums are arbitrary small. The case,  $\gamma(t_0) \neq x_j$  for all  $j = 1, 2, \dots$ , is similar.  $\square$

Next we define an auxiliary function  $S(y_0, \eta_0, t_0)$  which is non-zero if and only if there is  $j \in \mathbb{Z}_+$  such that  $\gamma_{y_0, \eta_0}(t_0) = x_j$ .

**Definition 3.** Let  $(y_0, \eta_0) \in T\mathbb{R}^n$  be such that  $y_0 \in \Omega^{int}$  and  $\|\eta_0\|_g = 1$ . We denote by  $\gamma(t; y_0, \eta_0) = \gamma_{y_0, \eta_0}(t)$  the geodesic on  $(\mathbb{R}^n, g)$  with  $\gamma(0) = y_0$ ,  $\dot{\gamma}(0) = \eta_0$ . Moreover, let  $w_\epsilon$  be a Gaussian beam of order  $N \geq 1 + n/4$  propagating along  $\gamma(t; y, \eta)$  backwards on time interval  $(T_0, t_0)$ . We define

$$S(y_0, \eta_0, t_0) := \lim_{\epsilon \rightarrow 0} \epsilon^{n/4} (h_0, w_\epsilon)_{\mathcal{E}'(\mathbb{R}^n \times (T_0, t_0))},$$

**Lemma 6.** Let  $(y_0, \eta) \in S\Omega$  and  $t_0 \in (0, T)$ . Then  $Lh_0$  for pseudorandom noise  $h_0$  and  $(\Omega, g|_\Omega)$ , given as a Riemannian manifold determine  $S(y_0, \eta_0, t_0)$ .

*Proof.* Let  $w_\epsilon$  be a Gaussian beam of order  $N \geq 1 + n/4$  propagating along the geodesic  $\gamma(\cdot; y_0, \eta_0)$  backwards on time interval  $(T_0, t_0)$ . We may choose the cut-off function  $\chi$  in the equation (18) so that  $w_\epsilon(t_0), \partial_t w_\epsilon(t_0) \in C_c^\infty(\Omega)$ . As  $g|_\Omega$  is known, we have by Remark 1 that the initial data  $w_\epsilon(t_0), \partial_t w_\epsilon(t_0)$  are known. Moreover, operators  $\text{Tr}_\Sigma \partial_t^j W_\Omega$ ,  $j = 0, 1$ ,  $\Sigma := \{t_0\} \times \Omega$ , are known. After choosing a suitable cut-off function  $\chi$  in Lemma 4 we have that the measurement  $Lh_0$  determines the distributional pairing  $(h_0, w_\epsilon)_{\mathcal{E}'(\mathbb{R}^n \times (T_0, t_0))}$ . Hence  $S(y_0, \eta_0, t_0)$  is determined.  $\square$

The implicit function theorem yields the following remark. Note that  $t_0 \in \mathbb{R}$  in the remark is not necessarily the first intersection time.

**Remark 2.** Let  $(y_0, \eta_0) \in S\mathbb{R}^n$  and  $t_0 \in \mathbb{R}$  satisfy

$$(\gamma(t_0; y_0, \eta_0), \dot{\gamma}(t_0; y_0, \eta_0)) \in \partial_\pm SM.$$

Then there are neighborhoods  $I \subset \mathbb{R}$  and  $U \subset S\mathbb{R}^n$  of  $t_0$  and  $(y_0, \eta_0)$  and a smooth map  $\ell : U \rightarrow I$  such that for  $t \in I$  and  $(y, \eta) \in U$

$$\gamma(t; y, \eta) \in \begin{cases} M, & \text{for } \pm t < \pm \ell(y, \eta), \\ \partial M, & \text{for } t = \ell(y, \eta), \\ \Omega, & \text{for } \pm t > \pm \ell(y, \eta). \end{cases}$$

We remind the reader that  $\tau(x, \xi)$ ,  $(x, \xi) \in T\mathbb{R}^n$ , is defined as the first intersection time with  $\partial M$ , that is

$$\tau(y_0, \eta_0) := \inf\{t \in (0, \infty]; \gamma(t; y_0, \eta_0) \in \partial M\}.$$

In the following, we use the Sasaki metric on the tangent bundle  $TM$ .

**Lemma 7.** *The first intersection times  $\tau : S\Omega \rightarrow (0, \infty]$  and  $\tau : \partial_- SM \rightarrow (0, \infty]$  are lower semi-continuous.*

*Proof.* Let us consider  $\tau$  on  $S\Omega$ . Let a sequence  $((y_j, \eta_j))_{j=1}^\infty \subset S\Omega$  converge to  $(y_0, \eta_0) \in S\Omega$  as  $j \rightarrow \infty$ . We denote  $\gamma_j(t) := \gamma(t; y_j, \eta_j)$  and  $\tau_j := \tau(y_j, \eta_j)$ .

We will show next that  $\liminf_{j \rightarrow \infty} \tau_j \notin (0, \tau_0)$ . Let  $t \in (0, \tau_0)$ . Then  $\gamma_0(t) \notin \partial M$  and

$$d(\gamma_0(t), \partial M) > 0.$$

Let  $j \in \mathbb{Z}_+$ . Suppose for a moment that  $\tau_j < \infty$ . Noting that  $\gamma_j$  is unit speed and  $\gamma_j(\tau_j) \in \partial M$ , we have

$$|t - \tau_j| \geq d(\gamma_j(t), \gamma_j(\tau_j)) \geq d(\gamma_j(t), \partial M).$$

If  $\tau_j = \infty$ , then  $|t - \tau_j| = \infty > d(\gamma_j(t), \partial M)$ .

The convergence  $\gamma_j(t) \rightarrow \gamma_0(t)$ , as  $j \rightarrow \infty$ , implies that for large  $j$

$$|t - \tau_j| \geq d(\gamma_0(t), \partial M)/2 > 0.$$

Hence  $\liminf_{j \rightarrow \infty} \tau_j \neq t$  for all  $t \in (0, \tau_0)$ .

Clearly  $\liminf_{j \rightarrow \infty} \tau_j \geq 0$ , and there is  $J \in \mathbb{Z}_+$  such that

$$\tau_j \geq d(y_j, \partial M) \geq d(y_0, \partial M)/2 > 0, \quad j \geq J.$$

Hence  $\liminf_{j \rightarrow \infty} \tau_j \neq 0$  and  $\liminf_{j \rightarrow \infty} \tau_j \geq \tau_0$ .

Let us consider  $\tau$  on  $\partial_- SM$ . Let a sequence  $((y_j, \eta_j))_{j=1}^\infty \subset \partial_- SM$  converge to  $(y_0, \eta_0) \in \partial_- SM$  as  $j \rightarrow \infty$ . We denote  $\gamma_j(t) := \gamma(t; y_j, \eta_j)$  and  $\tau_j := \tau(y_j, \eta_j)$ .

Repeating the above argument, we see that  $\liminf_{j \rightarrow \infty} \tau_j \notin (0, \tau_0)$ . Thus it is enough to show that  $\liminf_{j \rightarrow \infty} \tau_j \neq 0$ .

Remark 2 gives neighborhoods  $I \subset \mathbb{R}$  and  $U \subset S\mathbb{R}^n$  of zero and  $(y_0, \eta_0)$  and a map  $\ell : U \rightarrow I$  of boundary intersection times. We denote  $V := U \cap \partial_- SM$ . As  $\gamma(0; x, \xi) \in \partial M$  for  $(x, \xi) \in V$ , we have  $\ell = 0$  in  $V$ . In particular  $r := d(\ell(V), \mathbb{R} \setminus I) > 0$ . For large  $j$ ,  $(\gamma_j(0), \dot{\gamma}_j(0)) \in V$  and thus

$$\gamma_j(t) \in M, \quad t \in (0, r).$$

Hence  $\tau_j \geq r > 0$  for large  $j$ , and  $\liminf_{j \rightarrow \infty} \tau_j \geq \tau_0$ . □

We easily see the following continuity result for  $\tau$ .

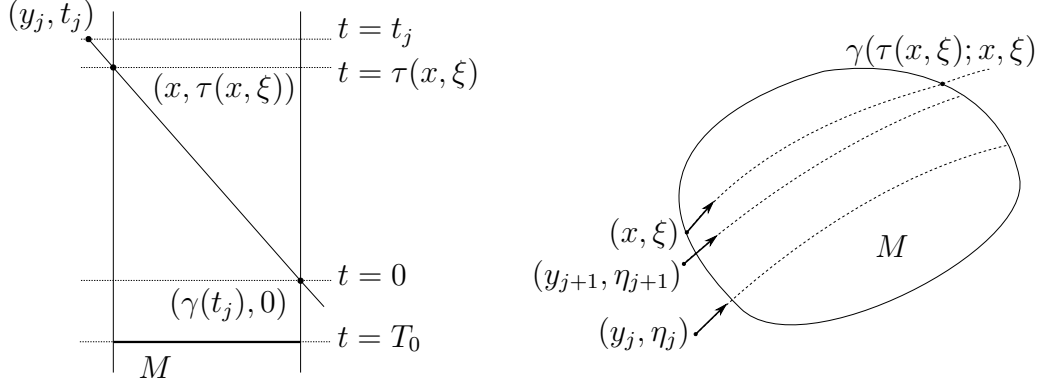


Figure 1: On left, trajectory of a Gaussian beam propagating along geodesic  $\gamma(t) := \gamma(t; y_j, \eta_j)$  backwards on time interval  $(T_0, t_j)$ . If  $S(y_j, \eta_j, t_j) \neq 0$ , then there is a point source at  $\gamma(t_j)$ . On right, a sequence  $(y_j, \eta_j) \in S\Omega$  converging to  $(x, \xi) \in \partial_- SM$  and trajectories of the corresponding geodesics.

**Lemma 8.** *Let  $((y_j, \eta_j))_{j=1}^\infty \subset S\Omega$  converge to  $(x, \xi) \in \partial_- SM$  in the Sasaki metric. Then  $\lim_{j \rightarrow \infty} \tau(y_j, \eta_j) = 0$ .*

**Theorem 2.** *Let  $(x, \xi) \in \partial_- SM$  and denote by  $J(x, \xi)$  the set of sequences  $((t_j; y_j, \eta_j))_{j=1}^\infty \subset (0, \infty) \times S\Omega$  for which*

$$\lim_{j \rightarrow \infty} (y_j, \eta_j) = (x, \xi), \quad \lim_{j \rightarrow \infty} t_j \in (0, \infty), \quad S(y_j, \eta_j, t_j) \neq 0.$$

*The function  $S : S\Omega \times (0, \infty) \rightarrow \mathbb{C}$  determines  $\tau : \partial_- SM \rightarrow (0, \infty]$  by the formula*

$$\tau(x, \xi) = \inf \left\{ \lim_{j \rightarrow \infty} t_j; ((t_j; y_j, \eta_j))_{j=1}^\infty \in J(x, \xi) \text{ for some } ((y_j, \eta_j))_{j=1}^\infty \subset S\Omega \right\}.$$

*Moreover, if  $\tau(x, \xi) < \infty$ , then there is a sequence  $((t_j; y_j, \eta_j))_{j=1}^\infty \in J(x, \xi)$  satisfying*

$$\tau(x, \xi) = \lim_{j \rightarrow \infty} t_j.$$

*Proof.* Let  $(x, \xi) \in \partial_- SM$  and  $((t_j; y_j, \eta_j))_{j=1}^\infty \in J(x, \xi)$ . Let us show, that  $\tau(x, \xi) \leq \lim_{j \rightarrow \infty} t_j$ . By Lemma 8,  $\tau_j := \tau(y_j, \eta_j) \rightarrow 0$  as  $j \rightarrow \infty$ . We define

$$\tilde{y}_j := \gamma(\tau_j; y_j, \eta_j), \quad \tilde{\xi}_j := \dot{\gamma}(\tau_j; y_j, \eta_j).$$

As  $S(y_j, \eta_j, t_j) \neq 0$ , we have

$$\gamma(t_j - \tau_j; \tilde{y}_j, \xi_j) = \gamma(t_j; y_j, \eta_j) \in \partial M.$$

As  $\lim_{j \rightarrow \infty} t_j > 0$  and  $\lim_{j \rightarrow \infty} \tau_j = 0$ , we have  $t_j - \tau_j > 0$  for large  $j$ . Thus  $\tau(\tilde{y}_j, \xi_j) \leq t_j - \tau_j$  for large  $j$ . Moreover,

$$\lim_{j \rightarrow \infty} (\tilde{y}_j, \xi_j) = (\gamma(0; x, \xi), \dot{\gamma}(0; x, \xi)) = (x, \xi).$$

In particular,  $(\tilde{y}_j, \xi_j) \in \partial_- SM$  for large  $j$ . Hence Lemma 7 gives

$$\lim_{j \rightarrow \infty} t_j = \lim_{j \rightarrow \infty} (t_j - \tau_j) \geq \liminf_{j \rightarrow \infty} \tau(\tilde{y}_j, \xi_j) \geq \tau(x, \xi).$$

In particular, we have proved the claim in the case  $\tau(x, \xi) = \infty$ .

Let us assume that  $\tau(x, \xi) < \infty$ . It is enough to show, that there is a sequence  $((t_j; y_j, \eta_j))_{j=1}^\infty \in J(x, \xi)$  satisfying  $\tau(x, \xi) = \lim_{j \rightarrow \infty} t_j$ . We denote

$$t_0 := \tau(x, \xi), \quad z := \gamma(t_0; x, \xi), \quad \zeta := -\dot{\gamma}(t_0; x, \xi).$$

We have

$$(x, \xi) = (\gamma(t_0; z, \zeta), -\dot{\gamma}(t_0; z, \zeta)).$$

As  $(x, \xi) \in \partial_- SM$ , Remark 2 gives neighborhoods  $I$  and  $U$  of  $t_0$  and  $(z, \zeta)$  and a map  $\ell : U \rightarrow I$  of boundary intersection times. After choosing local coordinates around  $z$  we may define

$$(y_j, \eta_j) := (\gamma(t_j; x_{k_j}, \zeta), -\dot{\gamma}(t_j; x_{k_j}, \zeta)),$$

where  $(x_{k_j})_{j=1}^\infty \subset U$  is a subsequence of the dense sequence of source points in (20) satisfying  $\lim_{j \rightarrow \infty} x_{k_j} = z$  and  $(t_j)_{j=1}^\infty \subset I$  satisfies

$$t_j > \ell(x_{k_j}, \zeta), \quad \lim_{j \rightarrow \infty} t_j = \ell(z, \zeta) = t_0.$$

Clearly  $((t_j; y_j, \eta_j))_{j=1}^\infty \in J(x, \xi)$  and

$$\lim_{j \rightarrow \infty} t_j = t_0 = \tau(x, \xi).$$

□

## 7 Determination of the scattering relation

In the next theorem we consider pseudorandom noise  $h_0(t, x)$  with coefficients

$$a_j = \lambda^{-\lambda^j},$$

with some  $\lambda > 1$  and make computations “modulo an error in  $A$ ”, where

$$A = \{-\lambda^j : j \in \mathbb{N}\}.$$

For this end, let  $m_A(s)$  be the real number  $r$  such that  $s = r + a$  where  $a \in A$  and  $r$  has the smallest possible absolute value. In the case when both  $r$  and  $-r$  satisfy this condition, we choose the positive value.

**Lemma 9.** *Let  $(y_0, \eta_0) \in S\Omega$ ,  $t_0 \in (0, T)$ , and suppose that  $S(y_0, \eta_0, t_0) \neq 0$ . Then there is a sequence  $((y_j, \eta_j))_{j=1}^\infty \subset S\Omega$  and  $(t_j)_{j=1}^\infty \subset (0, T)$  such that*

$$(y_j, \eta_j) \rightarrow (y_0, \eta_0), \quad t_j \rightarrow t_0, \quad S(y_j, \eta_j, t_j) \rightarrow 0, \quad \text{as } j \rightarrow \infty, \quad (21)$$

$$S(y_j, \eta_j, t_j) \neq 0.$$

*Suppose, moreover, that the coefficients of the pseudorandom noise  $h_0$  are  $a_j = \lambda^{-\lambda^j}$ . Then for any sequences  $((y_j, \eta_j))_{j=1}^\infty \subset T\mathbb{R}^n$  and  $(t_j)_{j=1}^\infty \subset (0, T)$  satisfying (21) we have that*

$$\lim_{j \rightarrow \infty} m_A(\log_\lambda |S(y_j, \eta_j, t_j)|) = \log_\lambda |u_0(t_0, \gamma(t_0); y_0, \eta_0) |g|^{1/2}(\gamma(t_0))|,$$

where  $\gamma(t) = \gamma(t; y_0, \eta_0)$  and  $u_0$  is defined as in (17).

*Proof.* We will use notation

$$\gamma_j(t) := \gamma(t; y_j, \eta_j), \quad z_j := \gamma_j(t_j), \quad S_j := S(y_j, \eta_j, t_j),$$

$$\beta_j := |u_0(t_j, z_j; y_j, \eta_j) |g|^{1/2}(z_j)|.$$

As  $S_0 \neq 0$ , we have that  $z_0 = x_j$  for some  $j = 1, 2, \dots$ . By continuity of the geodesic flow and density of  $(x_j)_{j=1}^\infty \subset \partial M$ , there exists a subsequence  $(x_{k_j})_{j=1}^\infty \subset (x_j)_{j=1}^\infty$  and sequences  $((y_j, \eta_j))_{j=1}^\infty \subset T\mathbb{R}^n$  and  $(t_j)_{j=1}^\infty \subset (0, T)$  such that

$$x_{k_j} \rightarrow z_0, \quad (y_j, \eta_j) \rightarrow (y_0, \eta_0), \quad t_j \rightarrow t_0, \quad \text{as } j \rightarrow \infty$$

and  $z_j = x_{k_j} \neq z_0$ . Then  $|S_j| = |a_{k_j}| \beta_j \neq 0$ . As  $x_{k_j} \neq z_0$  and  $x_{k_j} \rightarrow z_0$ , we have that  $k_j \rightarrow \infty$  and thus  $a_{k_j} \rightarrow 0$ . By (17) and continuity of the geodesic flow, it holds that  $\beta_j \rightarrow \beta_0 > 0$ . Hence  $S_j \rightarrow 0$ .

Next we use the assumption that  $a_j = \lambda^{-\lambda^j}$ . Let  $((y_j, \eta_j))_{j=1}^\infty \subset T\mathbb{R}^n$  and  $(t_j)_{j=1}^\infty \subset (0, T)$  satisfy (21). As  $S_j \neq 0$  we have that  $|S_j| = a_{k_j} \beta_j$  for some subsequence  $(a_{k_j})_{j=1}^\infty \subset (a_j)_{j=1}^\infty$ . As  $S_j \rightarrow 0$  we have that  $a_{k_j} \rightarrow 0$ . Moreover, sequence  $(\log_2 \beta_j)_{j=1}^\infty$  is bounded. This boundedness together with  $\log_\lambda a_{k_j} \in A$  and  $\log_\lambda a_{k_j} \rightarrow -\infty$  yield

$$m_A(\log_\lambda a_{k_j} + \log_\lambda \beta_j) = \log_\lambda \beta_j$$

for large  $j \in \mathbb{N}$ . Hence,

$$\lim_{j \rightarrow \infty} m_A(\log_\lambda |S_j|) = \lim_{j \rightarrow \infty} \log_\lambda \beta_j = \log_\lambda \beta_0.$$

□

**Theorem 3.** *If the coefficients of the pseudorandom noise  $h_0$  are  $a_j = \lambda^{-\lambda^j}$ , then the functions  $S : S\Omega \times (0, \infty) \rightarrow \mathbb{C}$  and  $\tau : \partial_- SM \rightarrow (0, \infty]$  determine  $D(\Sigma)$  and*

$$\gamma(\tau(x, \xi); x, \xi), \quad (x, \xi) \in D(\Sigma).$$

*Proof.* Clearly  $\tau$  on  $\partial_- SM$  determines  $D(\Sigma)$ . Let  $(x, \xi) \in D(\Sigma)$ . By Theorem 2 we may choose  $((t_j; y_j, \eta_j))_{j=1}^\infty \in J(x, \xi)$  such that  $\lim_{j \rightarrow \infty} t_j = \tau(x, \xi)$ . As  $S(y_j, \eta_j, t_j) \neq 0$ ,  $\gamma(t_j; y_j, \eta_j) = x_{k_j}$  for some subsequence  $(x_{k_j})_{j=1}^\infty$  of the sequence of source points. By Lemma 9 the function  $S$  determines

$$\frac{|S(y_j, \eta_j, t_j)|}{|u_0(t_j, x_{k_j}; y_j, \eta_j)| |g|^{1/2}(x_{k_j})} = a_{k_j}.$$

As  $a_j$ ,  $j \in \mathbb{Z}_+$ , are disjoint, this determines the index  $k_j$  and thus also the point  $x_{k_j}$ . Moreover

$$\lim_{j \rightarrow \infty} x_{k_j} = \lim_{j \rightarrow \infty} \gamma(t_j; y_j, \eta_j) = \gamma(\tau(x, \xi); x, \xi).$$

□

The following result follows from Remark 2.



**Lemma 10.** *Let us denote by  $X$  either  $S\Omega$  or  $\partial_-SM$ . Let  $(y_0, \eta_0) \in X$  satisfy*

$$\tau(y_0, \eta_0) < \infty, \quad \dot{\gamma}(\tau(y_0, \eta_0); y_0, \eta_0) \notin T_z \partial M,$$

*where  $z = \gamma(\tau(y_0, \eta_0); y_0, \eta_0)$ . Then there is a neighborhood  $V \subset X$  of  $(y_0, \eta_0)$  such that  $\tau = \ell$  in  $V$ , where  $\ell : U \rightarrow I$  is the map of boundary intersection times defined in Remark 2 for neighborhoods  $U \subset X$  and  $I \subset \mathbb{R}$  of  $(y_0, \eta_0)$  and  $\tau(y_0, \eta_0)$ . In particular,  $\tau$  is smooth in  $V$ .*

**Lemma 11.** *The set of  $(x, \xi)$  such that  $\gamma(\cdot; x, \xi)$  is transverse to  $\partial M$  is open and dense in*

$$\partial SM := \{(x, \xi) \in SM; x \in \partial M\}.$$

*Proof.* As  $\partial_-SM \cup \partial_+SM$  is open and dense in  $\partial SM$ , it is enough to show that the set of  $(x, \xi)$  such that  $\gamma(\cdot; x, \xi)$  is transverse to  $\partial M$  is open and dense in  $\partial_\pm SM$ . By the parametric transversality theorem, see [20, Thm. 3.2.7], the claim follows from the fact that the evaluation map

$$\begin{aligned} F^{ev} : \partial_\pm SM \times \mathbb{R} &\rightarrow \mathbb{R}^n \\ F^{ev} : (x, \xi, t) &\mapsto \gamma(t; x, \xi) \end{aligned}$$

is transverse to  $\partial M$ . □

**Lemma 12.** *Let  $(x_0, \xi_0) \in D(\Sigma)$ . Then there is a sequence  $((x_j, \xi_j))_{j=1}^\infty \subset D(\Sigma)$  such that  $\gamma(\cdot; x_j, \xi_j)$  is transverse to  $\partial M$  and*

$$\lim_{j \rightarrow \infty} (x_j, \xi_j) = (x_0, \xi_0), \quad \lim_{j \rightarrow \infty} \tau(x_j, \xi_j) = \tau(x_0, \xi_0).$$

*Proof.* We denote  $\tau_0 := \tau(x_0, \xi_0)$  and

$$(z_0, \zeta_0) := (\gamma(\tau_0; x_0, \xi_0), -\dot{\gamma}(\tau_0; x_0, \xi_0)).$$

Remark 2 gives a map of boundary intersection times  $\ell : U \rightarrow I$  for neighborhoods  $U \subset S\mathbb{R}^n$  and  $I \subset \mathbb{R}$  of  $(z_0, \zeta_0)$  and  $\tau_0$ . By Lemma 11 there is a sequence  $((z_j, \zeta_j))_{j=1}^\infty \subset SM \cap U$  converging to  $(z_0, \zeta_0)$  such that  $\gamma(\cdot; z_j, \zeta_j)$  is transverse to  $\partial M$ .

We define  $t_j := \ell(z_j, \zeta_j)$  and

$$(x_j, \xi_j) := (\gamma(t_j; z_j, \zeta_j), -\dot{\gamma}(t_j; z_j, \zeta_j)).$$

Then  $(x_j, \xi_j) \rightarrow (x_0, \xi_0)$  as  $j \rightarrow \infty$ . In particular, there is  $J \geq 1$  such that  $(x_j, \xi_j) \in \partial_- SM$  for  $j \geq J$ . By Lemma 7

$$\begin{aligned} \tau(x_0, \xi_0) &\leq \liminf_{j \rightarrow \infty} \tau(x_j, \xi_j) \leq \limsup_{j \rightarrow \infty} \tau(x_j, \xi_j) \\ &\leq \lim_{j \rightarrow \infty} \ell(z_j, \zeta_j) = \ell(z_0, \zeta_0) = \tau(x_0, \xi_0). \end{aligned}$$

□

**Lemma 13.** *Let  $(x_0, \xi_0) \in D(\Sigma)$  be such that  $\gamma(\cdot; x_0, \xi_0)$  is transverse to  $\partial M$ . Then there is  $(y_0, \eta_0) \in S\Omega$  lying on the geodesic  $\gamma(\cdot; x_0, \xi_0)$  and a neighborhood  $V \subset S_{y_0}\Omega$  of  $\eta_0$  such that the following conditions hold.*

(C1) *The map  $\eta \mapsto \tau(y_0, \eta)$  is smooth  $V \rightarrow (0, \infty)$ .*

(C2) *The map*

$$(x(\eta), \xi(\eta)) := (\gamma(\tau(y_0, \eta); y_0, \eta), \dot{\gamma}(\tau(y_0, \eta); y_0, \eta)) \quad (22)$$

*is smooth  $V \rightarrow D(\Sigma)$  and  $(x(\eta_0), \xi(\eta_0)) = (x_0, \xi_0)$ .*

(C3) *The map*

$$\tilde{\ell}(\eta) := \tau(x(\eta), \xi(\eta)) + \tau(y_0, \eta) \quad (23)$$

*is smooth  $V \rightarrow (0, \infty)$ .*

(C4) *There is a neighborhood  $W \subset \partial M$  of  $\gamma(\tau(x_0, \xi_0); x_0, \xi_0)$  such that*

$$\eta \mapsto \gamma(\tau(x(\eta), \xi(\eta)); x(\eta), \xi(\eta)) \quad (24)$$

*is a diffeomorphism  $V \rightarrow W$ .*

*Proof.* We denote  $\gamma(t) := \gamma(t; x_0, \xi_0)$  and  $z_0 := \gamma(\tau(x_0, \xi_0))$ . By remark 2  $\gamma(-t) \in \Omega$  for small  $t > 0$ . Moreover, the points that are conjugate to  $z_0$  along  $\gamma$  are discrete on  $\gamma$ , see e.g. [24].

Thus there is  $\tau_0 > 0$  such that

$$(y_0, \eta_0) := (\gamma(-\tau_0), \dot{\gamma}(-\tau_0))$$

is in  $S\Omega$ ,  $y_0$  is not conjugate to  $z_0$  along  $\gamma$ ,  $\tau(y_0, \eta_0) = \tau_0$  and

$$(\gamma(\tau_0; y_0, \eta_0), \dot{\gamma}(\tau_0; y_0, \eta_0)) = (x_0, \xi_0).$$

By Lemma 10 there is a neighborhood  $V_0 \subset S_{y_0}\Omega$  of  $\eta_0$  such that  $\eta \mapsto \tau(y_0, \eta)$  is smooth in  $V_0$ . Hence the function  $\eta \mapsto (x(\eta), \xi(\eta))$  maps  $\eta_0$  to  $(x_0, \xi_0)$  and is smooth in  $V_0$ . Moreover, this smoothness, transversality of  $\gamma(\cdot, x_0, \xi_0)$  and Lemma 10 imply that there is a neighborhood  $V_1 \subset V_0$  of  $\eta_0$  such that  $(x(\eta), \xi(\eta)) \in \partial_- SM$  and  $\eta \mapsto \tau(x(\eta), \xi(\eta))$  is smooth  $V_1 \rightarrow (0, \infty)$ . In particular,  $(x(\eta), \xi(\eta)) \in D(\Sigma)$  for all  $\eta \in V_1$ . We have shown that  $(y_0, \eta_0)$  and  $V_1$  satisfy (C1)-(C3).

We have

$$(\gamma(s; y_0, \eta), \dot{\gamma}(s; y_0, \eta))|_{s=t+\tau(y_0, \eta)} = (\gamma(t; x(\eta), \xi(\eta)), \dot{\gamma}(t; x(\eta), \xi(\eta))). \quad (25)$$

In particular,  $\gamma(\tilde{\ell}(\eta_0); y_0, \eta_0) = z_0$  and

$$\gamma(\tilde{\ell}(\eta); y_0, \eta) = \gamma(\tau(x(\eta), \xi(\eta)); x(\eta), \xi(\eta)) \in \partial M.$$

Moreover, as  $y_0$  is not conjugate to  $z_0$  along  $\gamma$ , there are neighborhoods  $V_2 \subset V_1$ ,  $I_0 \subset (0, \infty)$  and  $U_0 \subset \mathbb{R}^n$  of  $\eta_0$ ,  $\tilde{\ell}(\eta_0)$  and  $z_0$  such that  $(t, \eta) \mapsto \gamma(t; y_0, \eta)$  is a diffeomorphism  $V_2 \times I_0 \rightarrow U_0$ .

There is a neighborhood  $V \subset V_2$  of  $\eta_0$  such that  $\tilde{\ell}(V) \subset I_0$ . The graph of  $\eta \mapsto \tilde{\ell}(\eta)$  is an  $(n-1)$  dimensional submanifold on  $V \times I_0$ . Hence the diffeomorphism  $(t, \eta) \mapsto \gamma(t; y_0, \eta)$  maps it onto a  $(n-1)$  dimensional submanifold  $W$  of  $U_0$ . Moreover,  $z_0 \in W$  and  $W \subset \partial M$ . Thus  $W$  is a neighborhood of  $z_0$  in  $\partial M$ .  $\square$

**Lemma 14.** *Let  $(x_0, \xi_0) \in D(\Sigma)$  and  $(y_0, \eta_0) \in S\Omega$  satisfy conditions (C1)-(C4) of Lemma 13 for neighborhoods  $V \subset S_{y_0}\Omega$  and  $W \subset \partial M$  of  $\eta_0$  and  $z_0 := \gamma(\tau(x_0, \xi_0); x_0, \xi_0)$ . We denote by  $F : W \rightarrow V$  the inverse map of (24). Then*

$$\text{grad}_{\partial M}(\tilde{\ell} \circ F)|_{z=z_0} = \dot{\gamma}_{z_0}^\top, \quad (26)$$

where  $\tilde{\ell} : V \rightarrow (0, \infty)$  is the function (23) and  $\dot{\gamma}_{z_0}^\top$  is the orthogonal projection of  $\dot{\gamma}(\tau(x_0, \xi_0); x_0, \xi_0)$  into  $T_{z_0}\partial M$ .

*Proof.* Let  $\sigma : (-\epsilon, \epsilon) \rightarrow W$  be a smooth curve such that  $\sigma(0) = z_0$ . We define

$$\Gamma : (-\epsilon, \epsilon) \times \mathbb{R} \rightarrow \mathbb{R}^n, \quad \Gamma(s, t) := \gamma(t; y_0, F(\sigma(s))).$$

We denote  $\lambda := \tilde{\ell} \circ F \circ \sigma$  and  $\tilde{\ell}_0 := \tilde{\ell}(\eta_0)$ . By equation (25)

$$\begin{aligned} \Gamma(s, \lambda(s)) &= \gamma(\tau(x(\eta), \xi(\eta)); x(\eta), \xi(\eta))|_{\eta=F(\sigma(s))} = \sigma(s), \\ (\partial_t \Gamma)(0, \tilde{\ell}_0) &= \dot{\gamma}(\tilde{\ell}_0; y_0, \eta_0) = \dot{\gamma}(\tau(x_0, \xi_0); x_0, \xi_0). \end{aligned}$$

Hence

$$\dot{\sigma}(0) = \partial_s \Gamma(s, \lambda(s))|_{s=0} = (\partial_s \Gamma)(0, \tilde{\ell}_0) + (\partial_t \Gamma)(0, \tilde{\ell}_0) \lambda'(0).$$

The curve  $t \mapsto \Gamma(s, t)$  is a unit speed geodesic for all  $s \in (-\epsilon, \epsilon)$ . Hence

$$\begin{aligned} (\dot{\sigma}(0), \dot{\gamma}(\tau(x_0, \xi_0); x_0, \xi_0))_g &= ((\partial_s \Gamma, \partial_t \Gamma)_g + \lambda'(0)(\partial_t \Gamma, \partial_t \Gamma)_g)|_{s=0, t=\tilde{\ell}_0} \\ &= (\partial_s \Gamma, \partial_t \Gamma)_g|_{s=0, t=\tilde{\ell}_0} + \lambda'(0). \end{aligned} \quad (27)$$

We define

$$\mathcal{L}(s, l) := \int_0^l |\partial_t \Gamma(s, t)|_g dt, \quad (s, l) \in (-\epsilon, \epsilon) \times (0, \infty).$$

Then  $\mathcal{L}(s, l)$ ,  $s \in (-\epsilon, \epsilon)$  is the length of a unit speed geodesic on the interval  $[0, l]$ . Thus  $\mathcal{L}(s, l) = l$  for all  $s \in (-\epsilon, \epsilon)$ . We may derive an expression for  $\partial_s \mathcal{L}(s, l)|_{s=0}$  as in [36, Prop. 6.5]

$$\partial_s \mathcal{L}(s, l)|_{s=0} = \int_0^l (D_t \partial_s \Gamma, \partial_t \Gamma)_g dt|_{s=0}.$$

As  $t \mapsto \Gamma(s, t)$  is a geodesic,  $D_t \partial_t \Gamma(s, t) = 0$  and thus

$$\partial_t (\partial_s \Gamma, \partial_t \Gamma)_g = (D_t \partial_s \Gamma, \partial_t \Gamma)_g.$$

Moreover,  $\Gamma(s, 0) = y_0$  for all  $s \in (-\epsilon, \epsilon)$  and thus  $\partial_s \Gamma(s, 0) = 0$ . Hence

$$0 = \partial_s \mathcal{L}(s, l)|_{s=0} = \int_0^l \partial_t (\partial_s \Gamma, \partial_t \Gamma)_g dt|_{s=0} = (\partial_s \Gamma, \partial_t \Gamma)_g|_{s=0, t=l}, \quad l \in (0, \infty).$$

By (27) we have

$$\begin{aligned} (\dot{\sigma}(0), \gamma_{z_0}^\top)_g &= (\dot{\sigma}(0), \dot{\gamma}(\tau(x_0, \xi_0); x_0, \xi_0))_g \\ &= \lambda'(0) = \left\langle d(\tilde{\ell} \circ F)|_{z=z_0}, \dot{\sigma}(0) \right\rangle_{T_{z_0}^* \partial M \times T_{z_0} \partial M} \\ &= (\dot{\sigma}(0), \text{grad}_{\partial M}(\tilde{\ell} \circ F)|_{z=z_0})_g, \end{aligned}$$

for all smooth curves  $\sigma$  in  $W$  such that  $\sigma(0) = z_0$ , which proves the claim.  $\square$

**Theorem 4.** *The functions  $\tau : \partial_- SM \rightarrow (0, \infty]$  and*

$$z : D(\Sigma) \rightarrow \partial M, \quad z(x, \xi) := \gamma(\tau(x, \xi); x, \xi)$$

*together with the Riemannian manifold  $(\Omega, g|_\Omega)$  determine*

$$\dot{\gamma}(\tau(x, \xi); x, \xi), \quad (x, \xi) \in D(\Sigma).$$

*Proof.* The functions  $\tau$  and  $z$  on  $D(\Sigma)$  determine the set  $B$  of points  $(x_0, \xi_0) \in D(\Sigma)$  such that the conditions (C1)-(C4) of Lemma 13 hold for some  $(y_0, \eta_0) \in S\Omega$ .

Let  $(x_0, \xi_0) \in B$ . We denote  $\zeta_0 := \dot{\gamma}(\tau(x_0, \xi_0); x_0, \xi_0)$ . The map

$$\eta \mapsto z(x(\eta), \xi(\eta))$$

determines its local inverse. Hence  $\tau$  and  $z$  determine the function  $F$  of Lemma 14, and thus they determine  $\dot{\gamma}_{z_0}^\top$  by the formula (26). As  $\zeta_0$  is a unit vector

$$\zeta_0 = \dot{\gamma}_{z_0}^\top + (1 - |\dot{\gamma}_{z_0}^\top|^2)^{1/2} \nu_{z_0},$$

where  $\nu_{z_0}$  the unit exterior normal vector of  $\partial M$ . Hence  $\tau$  and  $z$  determine  $\zeta_0$  for all  $(x_0, \xi_0) \in B$ .

Let  $(x_0, \xi_0) \in D(\Sigma)$ . By Lemmata 12 and 13 there is a sequence  $((x_j, \xi_j))_{j=1}^\infty \subset B$  such that

$$\lim_{j \rightarrow \infty} (x_j, \xi_j) = (x_0, \xi_0), \quad \lim_{j \rightarrow \infty} \tau(x_j, \xi_j) = \tau(x_0, \xi_0).$$

Moreover, the functions  $\tau$  and  $z$  determine the set of such sequences and thus they determine

$$\lim_{j \rightarrow \infty} \dot{\gamma}(\tau(x_j, \xi_j); x_j, \xi_j) = \dot{\gamma}(\tau(x_0, \xi_0); x_0, \xi_0).$$

□

Theorems 2, 3 and 4 prove Theorem 1 formulated in the introduction.

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